Angular Momentum of a Brane-world Model

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Abstract

In this paper we discuss the properties of the general covariant angular momentum of a five-dimensional brane-world model. Through calculating the total angular momentum of this model, we are able to analyze the properties of the total angular momentum in the inflationary RS model. We show that the space-like components of the total angular momentum of the inflationary RS model are all zero while the others are non-zero, which agrees with the results from ordinary RS model.

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I. INTRODUCTION

Since the early works of Kaluza and Klein [1], the concept of extra spacetime dimensions has been broadly consumed by physicists. Recently the interests have been shifted from the traditional Kaluza-Klein type to the so-called "brane-world" picture which is inspired from the string theory. In brane models some fields (like SM fields) are localized at a brane while other fields (such as gravitation) can propagate in more dimensions. Numerous models of this kind have been proposed for different purposes, such as large extra dimensions models like the ADD scenario [2] and warped extra dimensions models like Randall-Sundrem (RS) models [3]. These models have different purposes ranging from solving the hierarchy problem to symmetry breaking.

On the other hand, the understanding of different types of conservation laws is very improtant in theories related to gravity, such as the conservation laws of energy momentum and angular momentum. In [4] Duan et al have proposed a generally covariant form of the conservation law of energy-momentum using the orthonormal frames, in which the energy-momentum is a covariant vector in Riemannian spacetime. It is generally covariant and is able to overcome the flaws in the expressions from Einstein and others. The usage of this form of energy-momentum has been conducted within the frame of RS model [5]. The result is generalized to more general conditions [6], which reflects the gauge hierarchy problem from a gravitational point of view. Following the similar procedure, Duan and Feng proposed a covariant conservation law of angular momentum [7], which is used to analyze the angular momentum conservation law in the RS model [8].

In this paper we analyze the angular momentum of a general brane-world model with one extra dimension. First in Section 2 we discuss the genreal setup of the model and some of the properties of the angular momentum in the RS model which are from [7]. Then in Section 3 we apply the method to calculate the angular momentum of this general brane model and a specific example — the inflationary RS model, which is a generalization of the original RS model. Finally in section 4 we present a conclusion of this paper.

II. THE SETUP AND THE ANGULAR MOMENTUM CONSERVATION

We follow the procedure of the RS model that we have a five-dimensional spacetime with two 3-branes in it. The fifth dimension which is labeled as y is compactified as $S^1/\mathbb{Z}_2$. The two 3-branes are located at the two fixed points of the orbifold $y=0$ and $y=\pi$. From a cosmological point of view, the 3-branes should be spatially homogeneous and isotropic. Furthermore, we assume that the usual three-dimensional space is also spatially flat. This gives us a general five-dimensional metric of the form [9]

$$ds^2 = -n^2(t,y)dt^2 + a^2(t,y)\delta_{ij}dx^i dx^j + b^2(t,y)dy^2 \tag{1}$$

The total action of this system is then

$$S = \int d^4x dy \sqrt{-g}\,\left[2M^3 R - \Lambda\right] + \sum_{i=1,2} \int d^4x \sqrt{-g^{(i)}}[\mathcal{L}_i - \Lambda_i] \tag{2}$$

where g_{MN} and R denote the five-dimensional metric and Ricci scalar respectively, Λ and Λ_i are the cosmological constants of that bulk and the branes, and $g^{(i)}_{\mu\nu}$ is the induced metric on the branes. The above Latin letters of M, N stand for the five-dimensional indices. The signature of g_{MN} is $(-++++)$. We have separated the gravitational part and the matter part of the action.

The construction of general relativity using the orthonormal frames, i.e. the vielbeins, has been discussed by many authors. The local frames represented by the vielbeins can be interpreted as a beautiful tool of expressing the equivalence principle. In [7,10] this method is used to analyze the covariant conservation law of energy-momentum and angular momentum. For energy-momentum conservation we need the general displacement transformations, while for angular momentum the conservation may be obtained using the local Lorentz invariance [8]. For our present model the form of the background metric of the spacetime manifold and the total action is a generalization of the original RS model, so from [8] it is shown that there exists a conserved total angular momentum tensor density with a superpotential

$$\partial_M(\sqrt{-g}j^M_{ab}) = 0 \tag{3}$$

$$j^M_{ab} = 2M^3\partial_N(\sqrt{-g}V^{NM}_{ab}) \tag{4}$$

Equation (3) is just the covariant conservation of the total angular momentum density, while the superpotential can be expressed as

$$V^{MN}_{ab} = e^M_a e^N_b - e^M_b e^N_a \tag{5}$$

where e^M_a is the vielbein. Therefore the superpotential and the angular momentum density have the following properties

$$V^{MN}_{ab} = -\,V^{NM}_{ab} = -V^{MN}_{ba} \tag{6}$$

$$j^M_{ab} = -j^M_{ba} \tag{7}$$

This total angular momentum density includes the spin density of the matter fields. It can be shown that the total conservative angular momentum can be obtained from

$$J_{ab} = \int_{\Sigma_t} e j^M_{ab} d\Sigma_M = 2M^3 \int_{\partial\Sigma_t} eV^{MN}_{ab} d\sigma_{MN} \tag{8}$$

where Σ_t is a cauchy surface of the spacetime manifold, e is the determinant of the vielbein, and $ed\Sigma_M$ is the covariant surface element of Σ_t with $d\Sigma_M = \frac{1}{4!}\,\epsilon_{MNOPQ}dx^N \wedge dx^O \wedge dx^P \wedge dx^Q$ and $d\sigma_{MN} = \frac{1}{3!}\,\epsilon_{MNOPQ}dx^O \wedge dx^P \wedge dx^Q$. The second part of Equation (8) is obtained from Gauss's law. On the cauchy surface Σ_t we have $dt = 0$, therefore the total angular momentum can be expressed as

$$J_{ab} = \int_{\Sigma_t} e j^t_{ab} dx^1 dx^2 dx^3 dy \tag{9}$$

We can see that this total angular momentum has the property $J_{ab} = -J_{ba}$.

III. THE ANGULAR MOMENTUM OF THE BRANE MODEL

Now let's analyze the angular momentum of our brane model. We can write the metric with orthonormal frames of this model as

$$ds^2 = -\hat{\theta}^0 \otimes \hat{\theta}^0 + \hat{\theta}^1 \otimes \hat{\theta}^1 + \hat{\theta}^2 \otimes \hat{\theta}^2 + \hat{\theta}^3 \otimes \hat{\theta}^3 + \hat{\theta}^4 \otimes \hat{\theta}^4 \tag{10}$$

so that the components of the orthonormal frames are

$$e_t^0 = n(t, y),\ e_{x^i}^i = a(t, y),\ e_y^4 = b(t, y) \tag{11}$$

Then from Equation (5) we can get the non-zero components of superpotential of our model

$$\begin{aligned} V_{0i}^{tx^i} &= V_{i0}^{x^it} = -V_{0i}^{x^it} = -V_{i0}^{tx^i} = e_0^t e_i^{x^i} = \frac{1}{na} \\ V_{04}^{ty} &= V_{40}^{yt} = -V_{04}^{yt} = -V_{40}^{ty} = e_0^t e_4^y = \frac{1}{nb} \end{aligned} \tag{12}$$

With the superpotential we can calculate the total angular momentum density. The non-vanashing components are

$$\begin{aligned} j_{04}^t &= -j_{40}^t = -6M^3a^2a' \\ j_{0i}^{x^i} &= -j_{i0}^{x^i} = -2M^3\partial_t(a^2b) \\ j_{04}^y &= -j_{40}^y = -6M^3a^2\dot{a} \end{aligned} \tag{13}$$

Finally we are able to obtain the two non-zero components of the total angular momentum of this brane model which is

$$J_{04} = -J_{40} = 6M_3\mathcal{V}\int nba^5a'dy \tag{14}$$

where $\mathcal{V}$ represents the volume of the three-dimensional space. We can see in this model all

space-like components of the total angular momentum are zero, which is a generalization of the results in [8]. If we plug in

$$a = n = e^{-kr|y|}, \qquad b = r \tag{15}$$

then we recover the asymptotic behavior in [8].

Now let's focus on one specific model of our kind — the inflationary RS model [11,6]. The metric is

$$ds^2 = \left(\frac{H_0 A_0}{k}\right)^2 \sinh^2(-kB_0|y| + c)[-dt^2 + e^{2H_0 t}\delta_{ij}\ dx^i dx^j] + B_0 dy^2 \tag{16}$$

which means that

$$a = \frac{A_0 H_0}{k}\ e^{H_0 t}\sinh(-kB_0|y| + c), \qquad n = \frac{H_0}{k}\sinh(-kB_0|y| + c), \qquad b = B_0 \tag{17}$$

where $H_0 = \frac{\dot{A}}{A}$, $k = \sqrt{\frac{-\Lambda}{24M^3}}$, and c is an integration constant which is related to other parameters by

$$k_1 = k\coth(c), \qquad -k_2 = k\coth\Big(-kB_0\pi + c\Big) \tag{18}$$

where $k_i = \Lambda_i/24M^3$. In [11] it is shown that the gauge hierarchy is a general property of both the inflationary RS model and the ordinary RS model, and in [6] the gauge hierarchy problem is analyzed from a gravitational point of view using the orthonormal frame method.

From Equation (16) we are able to get the non-zero components of the total angular momentum of this inflationary RS model

$$J_{04} = -J_{40} = 3M_3\mathcal{V}A_0^6(\frac{H_0}{k})^7\sinh(-kB_0\pi r)\sinh(-kB_0\pi r + 2c) \tag{19}$$

Therefore, in addition to the general property of our five-dimensional brane model that all

space-like components of the total angular momentum are zero, the non-space-like components are infinity, which is the result of gravity from the warped extra dimension. We can see that this is a general property of both the inflationary RS model and the ordinary RS model.

Let us now discuss some of the asymptotic behaviors of this result. If we set $r \to 0$, which means there is no extra dimension, we can see that Equation (19) becomes zero. This is obvious since in this case there is no effect from the gravity of the warped extra dimension. In [6] it is argued that if c is near $kB_0\pi r$, we can get an extremely large difference between these two energy densities. Here we can see that under this circumstance the general property of the total angular momentum we get here still holds.

IV. CONCLUSION

In conclusion, we have discussed the properties of the general covariant angular momentum of a five-dimensional brane-world model. Through calculating the total angular momentum of this model, we are able to analyze the properties of the total angular momentum in the inflationary RS model, whose static limit is the original RS model. We show that the space-like components of the total angular momentum of the inflationary RS model are all zero while the non-space-like components are infinity, which is the result of gravity from the warped extra dimension and agrees with the results from the ordinary RS model.

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